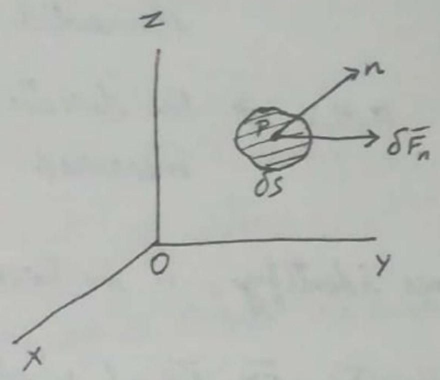
UNIT - V

VISCOUS FLOW

Stress components in a Real Fluid:



Let δs be a small rigid plane area inserted at a point P in a viscous fluid.

Cartesian co-ordinates (x, y, z) are referred to a set of fixed axes OX, OY, OZ. Suppose that δF_n is the force exerted by the moving fluid on one side of δs, the unit vector $\hat{n}$ being taken to specify the normal at P to δs on this side.

We know that in the case of an invisid fluid, δF_n is aligned with $\hat{n}$.

For a viscous fluid, however, frictional forces are called into play between the fluid and the surface so that δF_n will also have a component tangential to δs.

We suppose the Cartesian components of δF_n to be $[\delta F_{nx}, \delta F_{ny}, \delta F_{nz}]$,

so that $\delta F_n = \delta F_{nx} \vec{i} + \delta F_{ny} \vec{j} + \delta F_{nz} \vec{k}$

Then the components of stress parallel to the axes are defined to be p_{nx}, p_{ny}, p_{nz},

where, $$p_{nx} = \lim_{\delta s \to 0} \left(\frac{\delta F_{nx}}{\delta s}\right) = \frac{dF_{nx}}{ds}$$

$$p_{ny} = \lim_{\delta s \to 0} \left(\frac{\delta F_{ny}}{\delta s}\right) = \frac{dF_{ny}}{ds}$$

$$p_{nz} = \lim_{\delta s \to 0} \left(\frac{\delta F_{nz}}{\delta s}\right) = \frac{dF_{nz}}{\delta s}$$

In the components p_{nx}, p_{ny}, p_{nz}, ~~the first suffix ind~~

where $n \rightarrow$ the direction of the normal to the elemental plane δS.

$x, y, z \rightarrow$ the direction in which the component is measured.

we identify, $\hat{n}$ in turn with the unit vector $\bar{i}, \vec{j}, \vec{k}$ in $\overline{OX}, \overline{OY}, \overline{OZ}$ (which is achieved by suitably re-orientating δS), we obtain the following three sets of stress components :

$$p_{xx} \quad p_{xy} \quad p_{xz} \,;$$

$$p_{yx} \quad p_{yy} \quad p_{yz} \,;$$

$$p_{zx} \quad p_{zy} \quad p_{zz} \,;$$

where $p_{xx}, p_{yy}, p_{zz} \longrightarrow$ normal or direct stresses

$p_{xy}, p_{xz}, p_{yx}, p_{yz}, p_{zx}, p_{zy} \rightarrow$ Shearing stresses

For an Inviscid fluid

$$p_{xx} = p_{yy} = p_{zz} = -p \,;$$

$$p_{xy} = p_{xz} = p_{yx} = p_{yz} = p_{zx} = p_{zy} = 0.$$

Here, we regard the direct stresses as positive when they are tensile and negative when compressive, so that p is the hydrostatic pressure

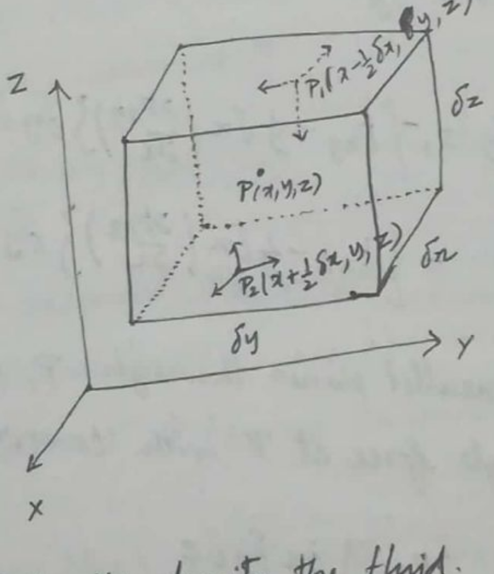
The matrix $\begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{bmatrix}$ is called the Stress matrix.

If its components are known, we can calculate the total forces on any area at any chosen point. The quantities p_{ij} ($i,j = x, y, z$) are called the components of the stress tensor whose matrix is of the above form.

Clearly p_{ij} is a second-order tensor.

Relation between Cartesian components of Stress:

Let us consider the motion of a small rectangular parallelopiped of viscous fluid, its centre being $P(x,y,z)$ and its edges of length $\delta x, \delta y, \delta z$ parallel to fixed Cartesian axes, as shown in the figure.



Let ρ be the density the fluid.

The mass $\rho\,\delta x\,\delta y\,\delta z$ of the fluid element remains constant and element is presumed to move along with the fluid.

In the diagram the points P_1, P_2 have coordinates $(x-\frac{1}{2}\delta x, y, z)$, $(x+\frac{1}{2}\delta x, y, z)$

At $P(x,y,z)$, the forces components parallel to $\overline{OX}, \overline{OY}, \overline{OZ}$ on the surface of area $\delta y \times \delta z$ through P and having $\hat{i}$ as unit normal are

$$[p_{xx}\,\delta y\delta z,\ p_{xy}\,\delta y\delta z,\ p_{xz}\,\delta y\, dz]$$

At $P_2(x+\frac{1}{2}\delta x, y, z)$

Since, $\hat{i}$ is the unit normal measured outwards from the fluid, the corresponding force components across the parallel plane of area $\delta y \times \delta z$ are

$$\left[\left\{p_{xx}+\frac{1}{2}\delta x\left(\frac{\partial p_{xx}}{\partial x}\right)\right\}\delta y\delta z,\ \left\{p_{xy}+\frac{1}{2}\delta x\left(\frac{\partial p_{xy}}{\partial x}\right)\right\}\delta y\delta z,\ \left\{p_{xz}+\frac{1}{2}\delta x\left(\frac{\partial p_{xz}}{\partial x}\right)\right\}\delta y\delta z\right].$$

For the parallel plane through $P_1(x-\frac{1}{2}\delta x, y, z)$

Since $-\hat{i}$ is the unit normal drawn outwards from the fluid element, the corresponding components are

$$\left[-\left\{p_{xx}-\frac{1}{2}\delta x\left(\frac{\partial p_{xx}}{\partial x}\right)\right\}\delta y\delta z, -\left\{p_{xy}-\frac{1}{2}\delta x\left(\frac{\partial p_{xy}}{\partial x}\right)\right\}\delta y\delta z,\ -\left\{p_{xz}-\frac{1}{2}\delta x\left(\frac{\partial p_{xz}}{\partial x}\right)\right\}\delta y\delta z\right].$$

The forces on the parallel planes through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial p_{xx}}{\partial x}, \frac{\partial p_{xy}}{\partial x}, \frac{\partial p_{xz}}{\partial x}\right]\delta x\,\delta y\,\delta z$$

together with couple whose moments (to the third order) are

$$\begin{cases} -p_{xz}\,\delta x\,\delta y\,\delta z \text{ about } Oy; \\ +p_{xy}\,\delta x\,\delta y\,\delta z \text{ about } Oz. \end{cases}$$

IIIly the pair of faces ⊥r to the y-axis give a force at P having components

$$\left[\frac{\partial p_{yx}}{\partial y}, \frac{\partial p_{yy}}{\partial y}, \frac{\partial p_{yz}}{\partial y}\right]\delta x\, \delta y\, \delta z$$

together with couple of moments

$$\begin{cases} -p_{yx}\,\delta x\, \delta y\, \delta z & \text{about } Oz; \\ +p_{yz}\,\delta x\, \delta y\, \delta z & \text{about } Ox; \end{cases}$$

The pair of faces perpendicular to the z-axis give a forces at P having components

$$\left[\frac{\partial p_{zx}}{\partial z}, \frac{\partial p_{zy}}{\partial z}, \frac{\partial p_{zz}}{\partial z}\right]\delta x\, \delta y\, \delta z$$

together with couples of moments

$$\begin{cases} -p_{zy}\,\delta x\, \delta y\, \delta z & \text{about } Ox; \\ +p_{zx}\,\delta x\, \delta y\, \delta z & \text{about } Oy. \end{cases}$$

~~The pair of faces perpendicular to the z-axis give a force at P having components~~

$$\cancel{\left[\frac{\partial p_{zx}}{\partial z}, \frac{\partial p_{zy}}{\partial z}, \partial p_{z}\right.}$$

Taking into account the surface forces on all six faces of the ~~cuboid~~, Parallelopiped, we observed that they reduce to a single force at P having components.

$$\left[\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right), \left(\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}\right), \left(\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}\right)\right]\delta x\, \delta y\, \delta z$$

together with a vector couple having cartesian components

$$[(p_{yz}-p_{zy}), (p_{zx}-p_{xz}), (p_{xy}-p_{yx})]\,\delta x\, \delta y\, \delta z.$$

Now suppose the external body forces acting at P are $[X, Y, Z]$ per unit mass, so that the total body force on the element has components. $[X, Y, Z]\,\rho\,\delta x\, \delta y\, \delta z$.

Let us take moments about $\vec{i}$ - direction through P.

Then we have

Total moment of forces = Moment of inertia about axis × Angular acceleration

i.e., $(p_{yz}-p_{zy})\delta x \delta y \delta z$ + terms of 4^{th} order in $\delta x, \delta y, \delta z$

= terms of 5^{th} order in $\delta x, \delta y, \delta z$

Thus, to the third order of smallness in $\delta x, \delta y, \delta z$, we obtain

$$(p_{yz}-p_{zy})\,\delta x\,\delta y\,\delta z = 0$$

Hence, as the considered fluid element becomes vanishingly small, we obtain

$$p_{yz}-p_{zy}=0 \quad \Rightarrow \boxed{p_{yz}=p_{zy}}$$

Ill^{ly}, we get

$$\boxed{p_{zx}=p_{xz}}, \quad \boxed{p_{xy}=p_{yx}}$$

Thus, the stress matrix is diagonally symmetric and contains only six unknowns.

In other words, we have proved that

$$\sigma_{ij}=\sigma_{ji}, \quad (i,j = x,y,z)$$

i.e., σ_{ij} is symmetric

In fact, σ_{ij} is a symmetric second order Cartesian tensor.

Translational Motion of Fluid Element:

Consider the Surface forces and body forces, the total force component in the i-direction acting on the fluid element ~~in the last section~~ at point $P(x,y,z)$ is

$$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z + X\rho\,\delta x\,\delta y\,\delta z$$

where (X, Y, Z) is the body force per unit mass and ρ being the density of the viscous fluid.

As the mass $\rho\,\delta x\,\delta y\,\delta z$ is considered constant, if $\bar{q}=(u,v,w)$ be the velocity of point P at time t, then the equation of motion in the $\hat{i}$-direction is

$$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z+\rho X\,\delta x\,\delta y\,\delta z=(\rho\,\delta x\,\delta y\,\delta z)\frac{du}{dt} \quad \to (1)$$

$\div$ by $\delta x\,\delta y\,\delta z$,

$$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)+\rho X=\rho\frac{du}{dt}$$

Now, $u=u(x,y,z,t)$ so that

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

Thus, (1) becomes.

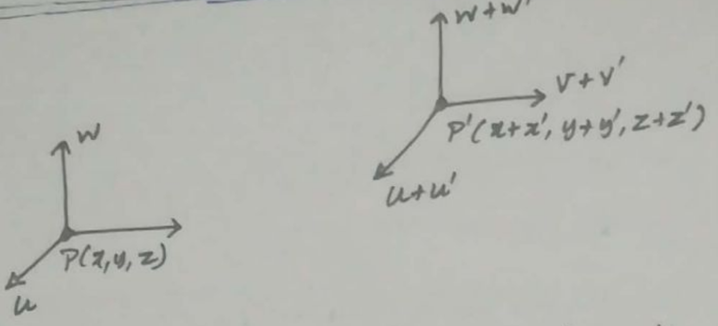
$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)\\
&\text{lly the equation of motion in } \hat{j} \text{ and } \hat{k} \text{ directions are}\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}=Y+\frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}\right)\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z+\frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)
\end{aligned}\right\}\to(2)$$

Eqn. (2) provide the equ. of motion of the fluid element at $P(x,y,z)$.

In tensor form, if the co-ordinates are (x_i), the velocity components (u_i), the body force components (X_i) $(i=1,2,3)$, the equations of motion may be written.

$$\frac{\partial u_i}{\partial t}+u_j u_{i,j}=X_i+\frac{1}{\rho}p_{ji,j} \qquad (i,j=1,2,3).$$

8.4 The Rate of Strain Quadric and Principal Stresses



Let $P(x,y,z)$, ~~and~~ $P'(x+x', y+y', z+z')$ are neighbouring points in a moving fluid at which the corresponding velocities have cartesian components $[u,v,w]$ $[u+u', v+v', w+w']$, all primed ~~quadric~~ quantities being small. Then from the rate of strain quadric for P is

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = \text{const.}$$

where $a = \frac{\partial u}{\partial x}$, $b = \frac{\partial v}{\partial y}$, $c = \frac{\partial w}{\partial z}$, $f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$

$g = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right)$ $h = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

Moreover the velocity increments due to pure strain are,

$$\delta \bar{q}_1 = A\,\delta\bar{r} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} ax' + hy' + gz' \\ hx' + by' + fz' \\ gx' + fy' + cz' \end{bmatrix}$$

When the rate of strain quadric is referred to its principal axes, it assumes the form

$$Ax''^2 + By''^2 + Cz''^2 = \text{const},$$

and the velocity increments due to pure strain referred to the principal axes are $[Ax'', By'', Cz'']$.

The stresses in the fluid elements are independent of its translation and rotation and depend solely on its distortion which is due to the rate of strain set up.

Since the stress matrix P & strain matrix A are symmetric there exist an orthogonal matrix H such that

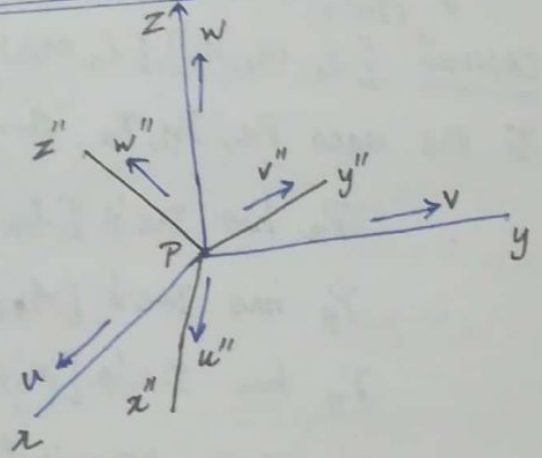
$$H^{-1}PH = \text{diagonal matrix}$$

$$H^{-1}AH = \text{diagonal matrix}$$

$$\therefore H^{-1}PH = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}$$ where p_1, p_2, p_3 are called principal stresses

and $$H^{-1}AH = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$ where a_1, a_2, a_3 are called principal strain.

Some Further properties of the Rate of strain Quadric



Let P be a point in the fluid moving with velocity $\bar{q}$.

Let P_x, P_y, P_z be a set of axes through P parallel to a given set of fixed Cartesian axes.

Let $[u, v, w]$ be the components of $\bar{q}$ with respect to these axes. Then, from the rate of strain quadric with centre at P has equation.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = \text{constant} \longrightarrow (1)$$

where $a = \left(\frac{\partial u}{\partial x}\right)_P$, $b = \left(\frac{\partial v}{\partial y}\right)_P$, $c = \left(\frac{\partial w}{\partial z}\right)_P$,

$$2f = \left(\frac{\partial w}{\partial y}\right)_P + \left(\frac{\partial v}{\partial z}\right)_P, \quad 2g = \left(\frac{\partial u}{\partial z}\right)_P + \left(\frac{\partial w}{\partial z}\right)_P, \quad 2h = \left(\frac{\partial v}{\partial x}\right)_P + \left(\frac{\partial u}{\partial y}\right)_P$$

Here the point (x, y, z) is a neighbouring point to P.

Now let P_x'', P_y'', P_z'' be the principal axes for this quadric Surface

Let $[u'', v'', w'']$ be the components of $\bar{q}$ along the directions of these principal axes.

Then the equation of the rate of strain quadric referred to its principal axes is

$$Ax''^2 + By''^2 + Cz''^2 = \text{constant} \longrightarrow (2)$$

where $A = \left(\frac{\partial u''}{\partial x''}\right)_p$, $B = \left(\frac{\partial v''}{\partial y''}\right)_p$, $C = \left(\frac{\partial w''}{\partial z''}\right)_p$.

If we suppose P_x'', P_y'', P_z'' to have Directions cosines (DC's) $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$, $[l_3, m_3, n_3]$ with respect to the axes Px, Py, Pz, then ~~Px has~~

Px has D.C's $[l_1, l_2, l_3]$

Py has D.C's $[m_1, m_2, m_3]$

Py has D.C's $[n_1, n_2, n_3]$

with respect to the principal axes.

Table of Directions Cosines

	Px	Py	Pz
P_x''	l_1	m_1	n_1
P_y''	l_2	m_2	n_2
P_z''	l_3	m_3	n_3

Clearly the following relations hold:

$l_1^2 + m_1^2 + n_1^2 = 1$, $l_2^2 + m_2^2 + n_2^2 = 1$, $l_3^2 + m_3^2 + n_3^2 = 1$

$l_1^2 + l_2^2 + l_3^2 = 1$, $m_1^2 + m_2^2 + m_3^2 = 1$, $n_1^2 + n_2^2 + n_3^2 = 1$

$l_1 m_1 + l_1 n_1 + m_1 n_1 = 0$, $l_2 m_2 + l_2 n_2 + m_2 n_2 = 0$, $l_3 m_3 + l_3 n_3 + m_3 n_3 = 0$

$l_1 l_2 + l_1 l_3 + l_2 l_3 = 0$, $m_1 m_2 + m_2 m_3 + m_1 m_3 = 0$, $n_1 n_2 + n_1 n_3 + n_2 n_3 = 0$

(12 equations)

we use the above properties to evaluate a, b, c, f, g, h in terms of A, B, C and the D.C's. 8.

Since

$$\frac{\partial}{\partial x} \equiv \frac{\partial x''}{\partial x} \cdot \frac{\partial}{\partial x''} + \frac{\partial y''}{\partial x} \cdot \frac{\partial}{\partial y''} + \frac{\partial z''}{\partial x} \cdot \frac{\partial}{\partial z''}$$

$$\equiv l_1 \frac{\partial}{\partial x''} + l_2 \frac{\partial}{\partial y''} + l_3 \frac{\partial}{\partial z''}$$

and $\quad u = l_1 u'' + l_2 v'' + l_3 w''$

$$a = \frac{\partial u}{\partial x} = l_1 \frac{\partial u''}{\partial x''} + l_2 \frac{\partial v''}{\partial y''} + l_3 \frac{\partial w''}{\partial z''}$$

$$= l_1 \left\{ l_1 \frac{\partial u''}{\partial x''} + l_2 \frac{\partial u''}{\partial y''} + l_3 \frac{\partial u''}{\partial z''} \right\} + l_2 \left\{ l_1 \frac{\partial v''}{\partial x''} + l_2 \frac{\partial v''}{\partial y''} + l_3 \frac{\partial v''}{\partial z''} \right\}$$

$$+ l_3 \left\{ l_1 \frac{\partial w''}{\partial x''} + l_2 \frac{\partial w''}{\partial y''} + l_3 \frac{\partial w''}{\partial z''} \right\}$$

$\because$ from (2), $\quad f = \frac{1}{2}\left(\frac{\partial v''}{\partial z''} + \frac{\partial w''}{\partial y''} \right) = 0$

$$g = \frac{1}{2}\left(\frac{\partial u''}{\partial z''} + \frac{\partial w''}{\partial x''} \right) = 0$$

$$h = \frac{1}{2}\left(\frac{\partial v''}{\partial x''} + \frac{\partial u''}{\partial y''} \right) = 0$$

$$\therefore \quad a = l_1^2 \frac{\partial u''}{\partial x''} + l_1 l_2 \underbrace{\left(\frac{\partial u''}{\partial y''} + \frac{\partial v''}{\partial x''} \right)}_{0} + l_1 l_3 \underbrace{\left(\frac{\partial u''}{\partial z''} + \frac{\partial w''}{\partial x''} \right)}_{0}$$

$$+ l_2^2 \frac{\partial v''}{\partial y''} + l_2 l_3 \underbrace{\left(\frac{\partial v''}{\partial z''} + \frac{\partial w''}{\partial y''} \right)}_{0} + l_3^2 \frac{\partial w''}{\partial z''}$$

$$a = l_1^2 \frac{\partial u''}{\partial x''} + l_2^2 \frac{\partial v''}{\partial y''} + l_3^2 \frac{\partial w''}{\partial z''}$$

|||ly $\quad b = m_1^2 \frac{\partial u''}{\partial x''} + m_2^2 \frac{\partial v''}{\partial y''} + m_3^2 \frac{\partial w''}{\partial z''}$

$$c = n_1^2 \frac{\partial u''}{\partial x''} + n_2^2 \frac{\partial v''}{\partial y''} + n_3^2 \frac{\partial w''}{\partial z''}$$

i.e)
$$\left. \begin{aligned} a &= l_1^2 A + l_2^2 B + l_3^2 C \\ b &= m_1^2 A + m_2^2 B + m_3^2 C \\ c &= n_1^2 A + n_2^2 B + n_3^2 C \end{aligned} \right\} \text{——(3)}$$

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Adding together equ. (3) gives

$$a+b+c = A+B+C$$

This Showing that $a+b+c$ is an invariant at any particular point for all orientations of axes.

This is obvious since

$$a+b+c = \text{div } q = \nabla \cdot q = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$A+B+C = \text{div } q = \nabla \cdot \bar{q} = \frac{\partial u''}{\partial x''} + \frac{\partial v''}{\partial y''} + \frac{\partial w''}{\partial z''}$$

and $2f = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$

$$= \left(\frac{\partial x''}{\partial z} \cdot \frac{\partial}{\partial x''} + \frac{\partial y''}{\partial z} \cdot \frac{\partial}{\partial y''} + \frac{\partial z''}{\partial z} \cdot \frac{\partial}{\partial z''}\right)(m_1 u'' + m_2 v'' + m_3 w'')$$

$$+ \left(\frac{\partial x''}{\partial y} \cdot \frac{\partial}{\partial x''} + \frac{\partial y''}{\partial y} \cdot \frac{\partial}{\partial y''} + \frac{\partial z''}{\partial y} \cdot \frac{\partial}{\partial z''}\right)(n_1 u'' + n_2 v'' + n_3 w'')$$

$$= \left(n_1 \frac{\partial}{\partial x''} + n_2 \frac{\partial}{\partial y''} + n_3 \frac{\partial}{\partial z''}\right)(m_1 u'' + m_2 v'' + m_3 w'')$$

$$+ \left(m_1 \frac{\partial}{\partial x''} + m_2 \frac{\partial}{\partial y''} + m_3 \frac{\partial}{\partial z''}\right)(n_1 u'' + n_2 v'' + n_3 w'')$$

$$= m_1 n_1 \frac{\partial u''}{\partial x''} + n_2 m_2 \frac{\partial v''}{\partial y''} + m_3 n_3 \frac{\partial w''}{\partial z''} + m_1 n_1 \frac{\partial u''}{\partial x''}$$

$$+ m_2 n_2 \frac{\partial v''}{\partial y''} + m_3 n_3 \frac{\partial w''}{\partial z''}$$ (Remaining terms are zero)

$$2f = 2(m_1 n_1 A + m_2 n_2 B + m_3 n_3 C)$$

$$\left.\begin{aligned} f &= m_1 n_1 A + m_2 n_2 B + m_3 n_3 C \\ g &= n_1 l_1 A + n_2 l_2 B + n_3 l_3 C \\ h &= l_1 m_1 A + l_2 m_2 B + l_3 m_3 C \end{aligned}\right\} \text{——} (4)$$

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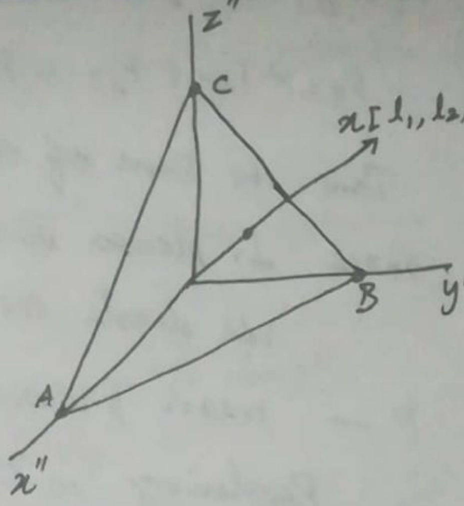
8.6 Stress Analysis in Fluid motion



Consider the plane $\perp$r to P_x, cuts the principal axes P_x'', P_y'', P_z'' of the rate of strain quadric in A, B, C to form a small tetrahedron of fluid PABC.

If δA denotes this area of the face ABC, then $l_1 \delta A$, $l_2 \delta A$, $l_3 \delta A$ are the areas of the faces PBC, PCA, PAB.

Since these last three are principal planes, using the notation of the principal stress P_1, P_2, P_3, it follows that the only forces on them are the normal forces $P_1 l_1 \delta A$, $P_2 l_2 \delta A$, $P_3 l_3 \delta A$.

The forces on the face ABC are $P_{xx}\delta A$, $P_{xy}\delta A$, $P_{xz}\delta A$ in the x, y, z direction (Since the plane is $\perp$r to P_x axis)

The equation of motion in the x-direction (mass x acceleration = Sum of forces at P)

$$\rho\, \delta A \frac{\partial u}{\partial t} = P_{xx}\delta A + (P_1 l_1 \delta A)(-l_1) + (P_2 l_2 \delta A)(-l_2) + P_3 l_3 \delta A(-l_3)$$

$$= P_{xx}\delta A - P_1 l_1^2 \delta A - P_2 l_2^2 \delta A - P_3 l_3^2 \delta A$$

The limit as the volume of the elements tends to zero, we get

$$\left.\begin{aligned} P_{xx} &= l_1^2 P_1 + l_2^2 P_2 + l_3^2 P_3 \\ \text{IIIly} \quad P_{yy} &= m_1^2 P_1 + m_2^2 P_2 + m_3^2 P_3 \\ P_{zz} &= n_1^2 P_1 + n_2^2 P_2 + n_3^2 P_3 \end{aligned}\right\} \longrightarrow (1)$$

Adding, we get

$$p_{xx} + p_{yy} + p_{zz} = p_1 + p_2 + p_3 \quad (= -3p, \text{say})$$

Thus, the sum of the normal stresses across any three $\perp r$ planes at a point is an invarient. we denote the sum by $3p$ so that

p — mean pressure at the point.

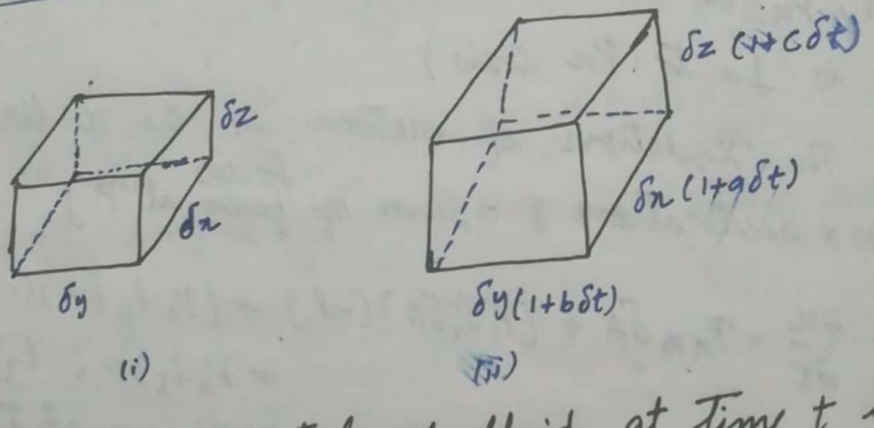
Resolving is the direction by the equation of motion.

$$\left.\begin{aligned} p_{xy} &= l_1 m_1 p_1 + l_2 m_2 p_2 + l_3 m_3 p_3 \\ p_{yz} &= m_1 n_1 p_1 + m_2 n_2 p_2 + m_2 n_3 p_3 \\ p_{zx} &= n_1 l_1 p_1 + n_2 l_2 p_2 + n_3 l_3 p_3 \end{aligned}\right\} \longrightarrow (2)$$

The eqn. (1) & (2) express the six distinct components of the stress matrix interms of the principal stress.

15M.

Relationship between stresses and rate of strain:



Consider a particle of fluid at time t in the shape of a rectangular parallelopiped of edges $\delta x, \delta y, \delta z$ $\parallel^{le}$ to fixed cartesian axis.

At time t the velocity component in the x-direction at the corner (x,y,z) of the box is u and so that at the corner $(x+\delta x, y, z)$ is

$$u + \left(\frac{\partial u}{\partial x}\right)\delta x$$

or $u + a\delta x$.

Thus at time $(t+\delta t)$, the edge δx has grown to length $\delta x + a\,\delta x\,\delta t$

$\because a\delta x$ is the relative velocity increase between its two ends.

|||ly the edges $\delta y, \delta z$ have grown to length $\delta y(1+b\delta t)$, $\delta z(1+c\delta t)$, respectively.

Thus the volumetric increment in the interval δt is $\delta x\,\delta y\,\delta z\,(1+a\,\delta t)\,(1+b\,\delta t)(1+c\,\delta t) - \delta x\,\delta y\,\delta z$

$$\simeq (a+b+c)\,\delta x\,\delta y\,\delta z\,\delta t$$

which gives a dilatation (or) volumetric strain in time δt of $(a+b+c)\,\delta t$.

Hence at time t, the rate of dilation is Δ

where $$\Delta = a+b+c = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}\bar{q}$$

This quantity has been seen to be invariant at each point of the fluid : its value is also $A+B+C$, in terms of principal rate of strain.

W.K.T, the equation of continuity ~~for~~ an incompressible fluid is -X-

$$\boxed{\Delta = 0}$$

for ~~an~~ compressible one -X-

$$\boxed{\Delta \neq 0}$$

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Case (i) Incompressible fluid

we Suppose that the principal stresses p_1, p_2, p_3 differ from their mean value $-p$ by quantities proportional to the rate of distortion A, B, C in the principal directions.

we write

$$\left.\begin{aligned} p_1 &= -p + 2\mu A, \\ p_2 &= -p + 2\mu B, \\ p_3 &= -p + 2\mu C, \end{aligned}\right\} \quad \text{—— (1)}$$

where μ is a constant

Case (ii) Compressible fluid

we have the additional effect of the rate of dilation Δ manifesting ~~..~~ itself equally in all directions. This effect we represent by adding to the R.H.S of each of the equation (1) the quantity $\lambda\Delta$, where λ is a constant,

$$(1) \Rightarrow \left.\begin{aligned} p_1 &= -p + 2\mu A + \lambda\Delta \\ p_2 &= -p + 2\mu B + \lambda\Delta \\ p_3 &= -p + 2\mu C + \lambda\Delta \end{aligned}\right\} \quad \text{—— (2)}$$

Adding together eqn. (2) and using $\Delta = A + B + C$, we find, Since $p_1 + p_2 + p_3 = -3p$ and $\Delta \neq 0$,

$$\lambda = -\frac{2}{3}\mu. \longrightarrow (3)$$

Eqn's (1) & (2) link principal stresses with principal rates of strain.

Next Evaluate non-principal stresses: $p_{xx}, p_{xy}, \dots p_{yz} \dots$ in terms of non-principal of strain.

W.K.T. $p_{xx} = l_1^2 p_1 + l_2^2 p_2 + l_3^2 p_3$

using the eqn. (2), we get

$\delta_{ij} = 1$ if $i = j$

$\delta_{ij} \neq 1$ if $i \neq j$

$u_{ij} = \frac{\partial u_i}{\partial x_j}$

$$p_{xx} = l_1^2(-p+2\mu A+\lambda\Delta) + l_2^2(-p+2\mu B+\lambda\Delta) + l_3^2(-p+2\mu C+\lambda\Delta)$$

$$= -p(l_1^2+l_2^2+l_3^2) + 2\mu(Al_1^2+Bl_2^2+Cl_3^2) + \lambda\Delta(l_1^2+l_2^2+l_3^2)$$

$$\left.\begin{aligned} p_{xx} &= -p+2\mu\frac{\partial u}{\partial x}+\lambda\Delta \\ \text{III}^{ly}\quad p_{yy} &= -p+2\mu\frac{\partial v}{\partial y}+\lambda\Delta \\ p_{zz} &= -p+2\mu\frac{\partial w}{\partial z}+\lambda\Delta \end{aligned}\right\} \longrightarrow (3)$$

where $\Delta = \nabla\cdot\bar{q}$, $\Delta = 0$ for incompressible flow

and $\Delta \neq 0$, $\lambda = -\frac{2}{3}\mu$ for compressible flow

$$p_{xy} = l_1m_1p_1 + l_2m_2p_2 + l_3m_3p_3$$

$$= l_1m_1(-p+2\mu A+\lambda\Delta) + l_2m_2(-p+2\mu B+\lambda\Delta) + l_3m_3(-p+2\mu C+\lambda\Delta)$$

$$= (-p+\lambda\Delta)(l_1m_1+l_2m_2+l_3m_3) + 2\mu(l_1m_1A+l_2m_2B+l_3m_3C)$$

$$\left.\begin{aligned} p_{xy} &= \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) = p_{yx} \\ \text{III}^{ly}\quad p_{yz} &= \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) = p_{zy} \\ p_{zx} &= \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) = p_{xz} \end{aligned}\right\} \longrightarrow (4)$$

which are true for compressible and incompressible fluids.

The equ's (3), (4) may be conviently combined in tensorial form. Thus if (x_i) denotes the Cartesian coordinates, (u_i) the velocity components $i=1,2,3$ then both set of equ. may be written

$$\boxed{p_{ij} = (\lambda\Delta - p)\delta_{ij} + \mu(u_{i,j}+u_{j,i})} \quad (i,j=1,2,3)$$

where $\Delta = u_{j,i}$, $p = -\frac{1}{3}p_{i,i}$, $\Delta = 0$ for incompressible flow, $\lambda = -\frac{2}{3}\mu$ for compressible pressible flow

8.8. The Coefficient of Viscosity and Laminar flow

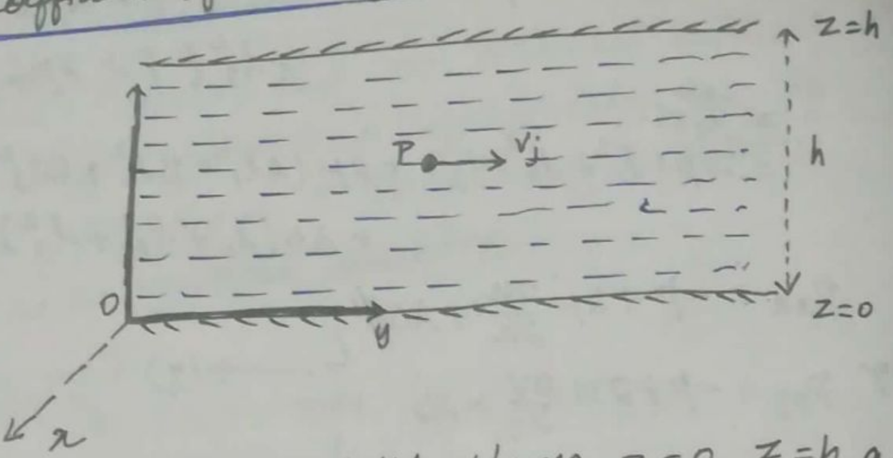

Fig. Shows two parallel planes $z=0$, $z=h$ a small distance h apart, the space between containing a thin film of viscous fluid.

The plane $z=0$ is fixed whilst the upper plane is given a constant velocity rightwards of amount Vj.
Then provided V is not excessively large, the layers of liquid in contact with $z=0$ are at rest. Whilst those in contact with $z=h$ are moving with velocity Vj.

i.e, there is no slip between fluid and either surface.

A velocity gradient is set up in the fluid between the planes.

At some point $P(x,y,z)$ in between the planes the fluid velocity will be vj where $0 < v < V$ and v is independent of x,y.

Thus when z is fixed, v is fixed,

i.e., the fluid moves in layers parallel to the two planes.
Such flow is termed laminar.

Due to the viscosity of the fluid there is friction between these parallel layers.

Experimental work shows that the shearing stress on the moving plane is proportional to V/h when h is sufficiently small. Thus we write this stress in the form $\mu' \frac{V}{h}$, where μ' is a constant called the coefficient of viscosity.

Now suppose $h \to 0$. Then the stress on the fixed plane

is $p_{zy} = \mu' \lim_{h \to 0} \left(\frac{V}{h}\right) = \mu' \frac{dv}{dz}$ $\longrightarrow (1)$

[Here the plane $z=0$, $z=h$ is $\perp$r to z-axis & x-axis the components are p_{xx}, p_{xy}, p_{xz} & p_{zx}, p_{zy}, p_{zz}].

p_{xy}, p_{xz}, p_{zy} are the shearing stresses.

W.K.T $p_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \mu \frac{\partial v}{\partial z} \rightarrow (2)$

$\because u=0, \quad v = v(z), w=0$

$p_{xz} = 0 \quad , p_{xy} = 0$

From (1) & (2), we get $\mu \frac{dv}{dz} = \mu' \frac{dv}{dz}$

$$\Rightarrow \mu = \mu'$$

This constant μ is called the coefficient of viscosity.

& $\nu = \mu/\rho$ is called ~~coefficient of~~ Kinematic viscosity.

From (1) we find the dimension of μ then,

$$[\mu] = \frac{[p_{zy}]}{[\frac{dv}{dx}]} = \frac{[MLT^{-2}]/L^2}{(LT^{-1})L^{-1}} = ML^{-1}T^{-1}$$

where M, L, T signify mass, length and time.

In aerodynamics, $\nu = \mu/\rho$

Thus $[\nu] = L^2T^{-1}$

most fluids, $\mu \rightarrow$ depends on the pressure & temperature

for gases, $\mu \rightarrow$ independent of the pressure but decreases with temperature

—x—

15 marks *:

The Navier - Stokes Equations of Motion of a Viscous fluid:

W.K.T the translation equation of motion in the form

$$\frac{du}{dt} = X + \frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right) \longrightarrow (1)$$

On Substituting $p_{xx} = -p + 2\mu\left(\frac{\partial u}{\partial x}\right) + \lambda\Delta$

$$p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$p_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

we : $\frac{\partial p_{xx}}{\partial x} = \frac{\partial}{\partial x}\left(-p + 2\mu\left(\frac{\partial u}{\partial x}\right) + \lambda\Delta\right)$ $\quad (\because \Delta = \nabla\cdot\bar{q} =$

$$= -\frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial}{\partial x}\Delta \longrightarrow (2)$$

$$\frac{\partial p_{yx}}{\partial y} = \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) = \mu\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\right) \longrightarrow (3)$$

$$\frac{\partial p_{zx}}{\partial z} = \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) = \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x\partial z}\right) \longrightarrow (4)$$

Adding (2) - (4) we get

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zx}}{\partial z} = -\frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial}{\partial x}\Delta$$
$$+ \mu\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x\partial z}\right)$$

$$= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \lambda\frac{\partial}{\partial x}(\nabla\cdot\bar{q})$$
$$+ \mu\frac{\partial}{\partial x}(\nabla\cdot\bar{q})$$

$$= -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \lambda\frac{\partial}{\partial x}(\nabla\cdot\bar{q}) + \mu\frac{\partial}{\partial x}(\nabla\cdot\bar{q})$$

$\therefore (1) \Rightarrow$ $$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u + \frac{\mu}{\rho}\frac{\partial}{\partial x}(\nabla\cdot\bar{q})$$
$$+ \frac{\lambda}{\rho}\frac{\partial}{\partial x}(\nabla\cdot\bar{q})$$

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$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \nu \frac{\partial \Delta}{\partial x} + \frac{\lambda}{\rho}\frac{\partial \Delta}{\partial x}$$

$\because \lambda = -\frac{2}{3}\mu$ for a compressible fluid and Sine $\Delta = 0$ for an incompressible fluid this equation may be written unambiguously for the two cases in the form

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \frac{1}{3}\nu \frac{\partial \Delta}{\partial x} \qquad (\because \nu = \frac{\mu}{\rho})$$

The Equation of motion for an incompressible fluid

$$\left.\begin{aligned} x: \frac{du}{dt} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \frac{1}{3}\nu\frac{\partial \Delta}{\partial x} \\ y: \frac{dv}{\partial t} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v + \frac{1}{3}\nu\frac{\partial \Delta}{\partial y} \\ z: \frac{dw}{\partial t} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w + \frac{1}{3}\nu\frac{\partial \Delta}{\partial z} \end{aligned}\right\} \text{——} (1)$$

Tensor form: $\frac{du_i}{dt} = X_i - \frac{1}{\rho} p_{,i} + \nu u_{i,jj} + \frac{1}{3}\nu \Delta_{,i}.$ —— (1')

writing $q = [x, y, z]$,

$F = [X, Y, Z]$

the vector form of equ. (1) is clearly

$$\frac{d\bar{q}}{dt} = F - \nabla \int \frac{dp}{\rho} + \nu \nabla^2 \bar{q} + \frac{1}{3}\nu \nabla(\nabla \cdot q) \longrightarrow (2)$$

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\wedge(\nabla\wedge\bar{q}) = F - \nabla\int\frac{dp}{\rho} + \frac{4}{3}\nu\nabla(\nabla\cdot q) - \nu\nabla\wedge(\nabla\wedge q) \longrightarrow (3)$$

where, $\frac{d\bar{q}}{dt} = \left(\frac{\partial \bar{q}}{\partial t}\right) + (\bar{q}\cdot\nabla)\bar{q}$

$= \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - q\wedge(\nabla\wedge q)$

$\nabla\wedge(\nabla\wedge q) = \nabla(\nabla\cdot q) - \nabla^2 q$

Eqn. of the forms (1), (2) & (3) are called the Navier Stokes equations of motion.

For incompressible flow,

Eqn's (2) & (3) give

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{q}$$

$$= F - \frac{1}{\rho}\nabla p + \nu \nabla \wedge (\nabla \wedge \bar{q}) \longrightarrow (4)$$

Flow is irrotational $\nabla \times \bar{q} = 0$

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p + \frac{4}{3}\nu \nabla . (\nabla . \bar{q})$$

Flow is irrotational & incompressible

$$(4) \Rightarrow \frac{d\bar{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

For inviscid flow

Letting $\nu \rightarrow 0$ then the flow is inviscid flow.

———x———